Temperature Rise in a Heat-Evolving Medium

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XII. *Temperature Rise in a Heat-Evolving Medium.*

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§ 1. *Introduction.*

Among the many problems of heat flow which are of importance to the physicist and the engineer, the question of a heat-evolving medium, *i.e.*, a medium evolving heat everywhere in its mass, has only recently found a place, and consequently it has as yet been but little considered from the theoretical standpoint. As exceptions to this general statement may be cited the astrophysical problem of heat generation in the interior of a star which has been considered by Eddington,* Milne,† and others, and the geophysical problem, treated by Jeffreys‡ and others, of the effect on the earth's cooling of the heat generated by radioactive substances in the earth's crust.

The increasing use, in engineering construction, of rapid hardening cement with its accompanying rapid heat evolution has caused the temperature rise due to heat of reaction to be no longer a negligible quantity, and it has thus become of considerable practical importance to obtain some means of estimating the magnitude of the temperature rise in large masses of concrete.

The theoretical problem thus presented of heat conduction in a chemically reacting mass in which the rate of heat evolution is a function of time, forms the subject of the present paper and, so far as the author is aware, no systematic treatment of this problem has hitherto been published.

In this connection it must be pointed out that the conditions of the problem are essentially different from both those previously mentioned, since in the astrophysical problem the mode of transfer of energy is by radiation, and not by conduction, while in the geophysical problem the rate of evolution of heat is assumed independent of the time. It is hoped, therefore, that the present paper will be both of theoretical interest in its treatment of a new problem, and also of practical utility in serving as a foundation on which to build a method of estimating the temperature rise in a mass of hydrating concrete.

* 'Z. Physik,' vol. 7, p. 351 (1921).

† 'Mon. Not. R. Astr. Soc.,' vol. 83, p. 118 (1923).

‡ "The Earth," Camb. Univ. Press (1929).

§ 2. *General Theory.*

We assume in our theory :—

(1) That the heat-evolving medium is homogeneous and that its conductivity and diffusivity are constant during the time considered.

(2) That the rate of heat evolution is the same at all points of the medium and is a function only of the time.*

(3) That at a boundary we have, either (*a*) a loss of heat obeying NEWTON's law, viz., rate of loss of heat proportional to excess of surface temperature above external temperature, or (*b*) the medium bounded by an ordinary homogeneous medium extending to infinity and of constant conductivity and diffusivity. A boundary of type (*a*) corresponds in practice to an air boundary and can also be extended to include a lagged surface,† while type (*b*) corresponds in practice to a surface bounded by earth, rock, etc.

(4) That the initial temperature of the heat-evolving medium is constant throughout the medium (being taken as zero without loss of generality), while the initial temperature of any bounding medium is also constant, but not necessarily equal to that of the heat-evolving medium.

In the heat-evolving medium let

T = temperature,
$\varepsilon(t)$ = rate of heat-evolution per unit volume,
κ = conductivity,
σ = heat capacity per unit volume,
x, y, z = rectangular Cartesian co-ordinates,
dS = surface element of any closed surface S drawn within the medium,
l, m, n = direction cosines of inward normal to dS,
$d\tau$ = volume element within S.

Then if we equate the rate of heat evolution within S to the rate of increase of heat within S plus the rate of loss of heat across S we obtain

$$\iiint \varepsilon(t)\, d\tau = \iiint \sigma \frac{\partial T}{\partial t}\, d\tau + \iint \kappa \left(l \frac{\partial T}{\partial x} + m \frac{\partial T}{\partial y} + n \frac{\partial T}{\partial z} \right) dS.$$

Transforming the surface integral by GREEN's theorem we obtain

$$\iiint \left(\varepsilon(t) - \sigma \frac{\partial T}{\partial t} + \kappa \nabla^2 T \right) d\tau = 0,$$

* Physically, it would be preferable to extend the assumption to include the effect of temperature on the rate of heat evolution, but unfortunately the equations of heat flow then become intractable. For hydrating concrete the author has, however, circumvented this difficulty by using the results of the present paper combined with an empirical correction for the temperature coefficient of velocity of reaction.

† FOX, E. N., " Two Problems arising in Practical Applications of Heat Theory " (*in course of publication*).

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and since this is true for the volume enclosed by any surface S within the medium, we have

$$\varepsilon(t) = \sigma \frac{\partial T}{\partial t} - \kappa \nabla^2 T. \tag{1}$$

The maximum possible temperature rise will occur when there is no heat loss from the boundaries, and denoting this temperature rise by T_M (a function of time t) we then have from (1)

$$\varepsilon(t) = \sigma \frac{dT_M}{dt}, \tag{2A}$$

whence

$$T_M = \frac{1}{\sigma}\int_0^t \varepsilon(t)\,dt. \tag{2B}$$

We may eliminate $\varepsilon(t)$ from (1) and (2A) and obtain for our fundamental equation

$$h^2 \nabla^2 T = \frac{\partial T}{\partial t} - \frac{dT_M}{dt}, \tag{3}$$

where $h^2 = \frac{\kappa}{\sigma}$ = diffusivity of the heat-evolving medium.

We shall in this paper use equation (3) rather than equation (1) and thus express the results in terms of T_M rather than in terms of $\varepsilon(t)$ since in practice probably the simplest method of measuring heat evolution is by recording the temperature rise in a small specimen kept in an adiabatic calorimeter* and then $T_M(t)$ is a directly recorded quantity while $\varepsilon(t)$, if required, must be obtained by differentiation.

It would therefore appear better to have solutions in which the recorded curve can be directly used, rather than solutions which necessitate first differentiating the recorded curve. In any case the solutions given here can always be expressed in terms of $\varepsilon(t)$ by means of equation (2B).

Equation (3) governs the heat flow within the mass while at a boundary we have

(*a*) If NEWTON's law holds

$$-\kappa \frac{\partial T}{\partial n} = E\{T - T_A(t)\}, \tag{4}$$

where ∂n is in direction of outward normal to the surface, E is the emissivity of the surface, and $T_A(t)$—a known function of t—is the external temperature outside the surface.

(*b*) If the heat-evolving medium is bounded by an ordinary medium of temperature T_1, diffusivity h_1^2, and conductivity κ_1 we then have, firstly, the conduction equation for the bounding medium,

$$h_1^2 \nabla^2 T_1 = \frac{\partial T_1}{\partial t}, \tag{5}$$

* DAVEY, 'Concr. Constr. Eng.,' vol. 26, p. 572 (1931).

and, secondly, at the boundary between the media

$$\left.\begin{aligned} T &= T_1 \\ -\kappa \frac{\partial T}{\partial n} &= -\kappa_1 \frac{\partial T_1}{\partial n} \end{aligned}\right\}, \tag{6}$$

where ∂n is in the direction of the outward normal to the heat-evolving medium.

Finally, on account of our fourth assumption, we have at $t = 0$ the initial conditions

$$T = 0, \tag{7}$$

throughout the heat-evolving medium and

$$T_1 = D, \tag{8}$$

throughout the bounding medium, D being a constant. Equation (8) also involves implicitly the condition for all t

$$T_1 \rightarrow D, \tag{9}$$

as we tend to infinity in the bounding medium.

Before attempting to solve the preceding equations in any particular case we will first of all derive some general results.

In equations (3), (4), and (7), let

$$\left.\begin{aligned} T &= T_M - T' \\ T_A &= T_M - T'_A \end{aligned}\right\}. \tag{10}$$

Equations (4) and (7) remain the same except for the addition of accents to the symbols while equation (3) becomes

$$h^2 \nabla^2 T' = \frac{\partial T'}{\partial t}. \tag{11}$$

This is of the same form as the conduction equation for an ordinary medium, and we therefore see that if all the boundaries are of type (a) the solution of our present problem can be simply deduced from that of the problem of an ordinary medium in the presence of varying external temperatures, and this latter solution can in turn be deduced, by DUHAMEL's theorem,* from the solution of the extensively treated problem of constant external temperatures.

Consider now the case when one of the boundaries is of type (b) the remainder being of type (a). Then if we put in our general equations

$$\left.\begin{aligned} T &= T_{M_1} - T' \\ T_M &= T_{M_1} - T'_M \\ T_A &= T_{M_1} - T'_A \\ T_1 &= T_{M_1} - T'_1 \\ D &= -D' \end{aligned}\right\}, \tag{12}$$

* CARSLAW, "The Conduction of Heat," p. 17. London, Macmillan & Co. (1921).

where T_{M_1} to be interpreted later, is a function of t only and vanishes when $t = 0$, we find that the only equations changed in form are equations (5) and (9) which become

$$h_1^2\nabla^2 T'_1 = \frac{\partial T'_1}{\partial t} - \frac{dT_{M_1}}{\partial t}$$

$$T'_1 \rightarrow D' + T_{M_1}.$$

Hence, if we interpret T_{M_1} as the adiabatic temperature rise corresponding to a heat evolution in the bounding medium, and if we drop the accents to the symbols in the transformed equations, then these latter become the general equations when both media are heat-evolving. Thus, the solution of the more general problem where the bounding medium is also heat evolving may be simply deduced from the solution of the original problem of only one heat-evolving medium by substituting according to (12) for T, T_M, etc., and then dropping the accents to the symbols.

Furthermore, by putting $T_M = 0$ in the general solution so deduced we can obtain the solution where the roles of heat-evolving medium and of bounding medium have been interchanged.

This result may be generalized by saying that if we have n media such that the boundary condition between any two of them is of similar form to equation (6) while any other boundaries of the media are of type (a), then the solution for the most general case where all the media are heat-evolving (with different T_M) may be deduced, by a transformation similar to equation (12) from the solution of any one of the n cases where only $n - 1$ of the media are heat-evolving.

Returning to the case of only one heat-evolving medium let us put in equations (3) to (9)

$$\left.\begin{aligned} T &= T_M - T' - T'' \\ T_A &= T_M - T'_A \\ T_1 &= - T'_1 \\ D &= - D' \end{aligned}\right\}, \tag{13}$$

where T'' is defined by the equations

$$\left.\begin{aligned} & h^2\nabla^2 T'' = \frac{\partial T''}{\partial t} \\ \text{and} \quad & - \kappa\frac{\partial T''}{\partial n} = ET'' \qquad \text{at boundary } (a) \\ & T'' = T_M \qquad \text{at boundary } (b) \\ & T'' = 0 \qquad \text{when } t = 0 \end{aligned}\right\}. \tag{14}$$

Then equation (3) is transformed to equation (11), equations (4), (5), (7), (8), and (9) have accents added to the temperature symbols, while equation (6) becomes

$$T' = T'_1,$$

$$\kappa\frac{\partial T'}{\partial n} = \kappa_1\frac{\partial T'_1}{\partial n} - \kappa\frac{\partial T''}{\partial n}.$$

Now equation (14) shows that T'' is the solution of the well-known problem of a non-heat-evolving medium where at the boundaries, either the temperature is a given function of t or NEWTON'S law holds, the external temperature being zero. Hence, regarding T'' as known, $-\kappa\,\partial T''/\partial n$ at any boundary will be a known function of t and the problem of finding T' is similar to the original problem except that instead of a known heat evolution within the medium we have heat sources of known strength $-\kappa\,\partial T''/\partial n$ per unit area at each boundary of type (b). The general case of any number of media, n of which are heat-evolving can be similarly reduced to the consideration of, firstly, n solutions of type T'' each involving consideration of only one medium, and, secondly, where none of the media are heat-evolving, but instead there are heat sources at the common boundaries.

Having thus shown how the general problem can be made to depend on certain problems of heat flow in non-heat-evolving media, it must be pointed out that unless the solutions of these latter problems are already known and to hand, it is simpler in practice to seek a direct solution of the fundamental equations (3) to (9), and this latter method will therefore be adopted in the present paper.

Now where the flow of heat is three- or two-dimensional, the solution of our equations, if one can be obtained, usually takes the form of a triply or doubly infinite series respectively, and is thus far too cumbersome for application in practice, and we shall therefore consider only the six most important cases of one-dimensional flow. Fortunately, in practice, the flow of heat is mainly in one direction and we can use the appropriate one-dimensional solution, and if necessary incorporate in it, in the manner described elsewhere (FOX, *loc. cit.*), approximate corrections for the small transverse heat losses.

In the mathematics which follows we shall make use of the operational calculus* since this gives probably the simplest and most direct method of solving our equations, and furthermore lends itself readily to the expressing of the solution in more than one form. For this latter reason, *i.e.*, in order to deduce alternative forms, the solutions of § 3–5 will be derived directly by the operational calculus instead of by the application, to known solutions, of the transformation given by equation (10). Since the method of solution for the cylinder, slab, and sphere is essentially the same, it will only be expounded fully for the cylinder, which has been chosen, rather than the slab, for the more detailed exposition as the algebra is less complicated.

§ 3. *Temperature Rise in Circular Cylinder.* NEWTON'S *Boundary Law.*

The cylinder is assumed to be either infinitely long or else perfectly insulated at ends and thus the only flow of heat occurs in a radial direction. At the cylindrical boundary the heat losses are assumed to obey NEWTON'S law.

* JEFFREYS, 'Camb. Tract. Math.,' No. 23 (1927).

For this case our general equations simplify to :—

$$h^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} = \frac{\partial T}{\partial t} - \frac{dT_M}{dt}, \tag{15}$$

$$-\kappa \frac{\partial T}{\partial r} = E(T - T_A), \qquad (r = a) \tag{16}$$

$$T = 0 \text{ when } t = 0, \qquad (0 \leqslant r \leqslant a), \tag{17}$$

where a is the radius of the cylinder. Putting $\frac{\partial}{\partial t} = p = h^2 q^2$ then in the notation of the operational calculus (15) becomes

$$h^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} = p(T - T_M) = h^2 q^2 (T - T_M). \tag{18}$$

This we regard as an ordinary differential equation in r which, combined with (17) will give T as a function of r.

The solution of (18) which is finite at the origin $r = 0$ is given by

$$T = T_M - AI_0(qr).$$

Hence applying (16) we have

$$\kappa A q I_1(qa) = E(T_M - AI_0(qa) - T_A).$$

Putting

$$Ea/\kappa = \eta,$$

we obtain

$$A = \frac{\eta(T_M - T_A)}{\eta I_0(qa) + qaI_1(qa)},$$

and thus our solution becomes

$$T = T_M - \frac{\eta I_0(qr)(T_M - T_A)}{\eta I_0(qa) + qaI_1(qa)}$$
$$= T_M - \chi(p)(T_M - T_A) \tag{19}$$

which defines the operator $\chi(p)$.

In this solution T_M and T_A are known empirical functions of t, and in order to interpret (19) subject to (17) we use the relation valid for $t \geqslant 0$

$$T_M(t) - T_A(t) = -\int_{\lambda=0}^{\infty} \{T_M(\lambda) - T_A(\lambda)\} \, d\mathsf{H}(t - \lambda)$$
$$= \int_0^{\infty} \left[\frac{d}{d\lambda}\{T_M(\lambda) - T_A(\lambda)\}\right] \mathsf{H}(t - \lambda)\, d\lambda - T_A(0), \tag{20}$$

where

$$\left.\begin{aligned} \mathsf{H}(t) &= 1 \qquad t \geqslant 0 \\ &= 0 \qquad t < 0 \end{aligned}\right\} \text{ is HEAVISIDE's " unit " function.}$$

Substituting in (19) we obtain

$$T = T_M(t) - \int_0^\infty \left[\frac{d}{d\lambda}\{T_M(\lambda) - T_A(\lambda)\}\right] \chi(p)\, \mathsf{H}(t-\lambda)\, d\lambda + \chi(p)\, T_A(0). \qquad (21)$$

Now by BROMWICH's rule

$$\chi(p)\,\mathsf{H}(t-\lambda) = \frac{1}{2\pi i}\int_L \frac{e^{p(t-\lambda)}\chi(p)\,dp}{p}$$
$$= \frac{1}{2\pi i}\int_L \frac{e^{p(t-\lambda)}\,\eta I_0(qr)\,dp}{\{\eta I_0(qa) + qaI_1(qa)\}\,p},$$

where L is line from $c - i\infty$ to $c + i\infty$, $(c > 0)$ in the plane of the complex variable p.

Consider first the poles of the integrand, which is a single valued function of p; these will occur at $p = 0$ and at the roots of the denominator of $\chi(p)$, *i.e.*, at the roots of

$$\eta I_0(qa) + qaI_1(qa) = 0.$$

If we put $qa = i\phi$, this becomes

$$\eta J_0(\phi) - \phi J_1(\phi) = 0.$$

Now it is easily shown* that this equation has no roots except when ϕ is real and that it then possesses an infinite number of roots $\phi_1, \phi_2, \ldots, \phi_n, \ldots$, forming an increasing sequence in which $\phi_n \to \infty$ as $n \to \infty$. Hence the poles of $\chi(p)$ are given by

$$p = h^2q^2 = -\frac{h^2\phi_n^2}{a^2}, \qquad (n = 1, 2, \ldots, \infty),$$

i.e., they all lie on the negative real axis.

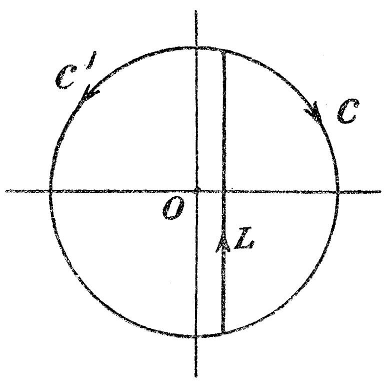

FIG. 1.—Complex p-plane.

(i) Let $t - \lambda < 0$.—Then we consider $\int \frac{e^{p(t-\lambda)}\chi(p)\,dp}{p}$ taken round the contour formed by L from $c - i$R to $c + i$R and the large semi-circle C, radius R, to the right of L.

* WATSON, "Theory of Bessel Functions," p. 482, Camb. Univ. Press (1922).

Since there are no poles of integrand inside the contour the contour integral is variable. Hence if we let $R \to \infty$ we get

$$\int_L + \underset{R\to\infty}{\mathrm{Lt}} \int_C = 0.$$

But by JORDAN's lemma the second term is zero and we therefore have

$$\int_L \frac{e^{p(t-\lambda)}\chi(p)\,dp}{p} = 0, \qquad t-\lambda < 0.$$

(ii) Let $t - \lambda > 0$.—In this case we take the integral round the contour formed by part of L and the large semicircle C′ radius R to the left of L, in which R is so chosen that the semicircle cuts the negative real axis at a finite distance from the poles of the integrand. We then have that the contour integral is equal to $2\pi i$ times the sum of the residues at the poles of the integrand within the contour. If we now let $R \to \infty$ the integral round the semicircle $\to 0$, and we get,

$$\frac{1}{2\pi i}\int_L \frac{e^{p(t-\lambda)}\chi(p)\,dp}{p} = \Sigma \text{ residues at poles of integrand.}$$

At $p = 0$, residue $= 1$.

$$\text{At } p = -\frac{h^2\phi_n^2}{a^2}, \text{ residue} = -\frac{2\eta J_0\left(\frac{\phi_n r}{a}\right)e^{-\frac{h^2\phi_n^2}{a^2}(t-\lambda)}}{(\eta^2+\phi_n^2)\,J_0(\phi_n)},$$

after simplification.

We thus have

$$\chi(p)\,\mathrm{H}(t-\lambda) = 0 \qquad (t<\lambda)$$

$$= 1 - \sum_{n=1}^{\infty}\frac{2\eta J_0\left(\frac{\phi_n r}{a}\right)e^{-\frac{h^2\phi_n^2}{a^2}(t-\lambda)}}{(\eta^2+\phi_n^2)\,J_0(\phi_n)} \qquad (t>\lambda). \tag{22}$$

For the $T_A(0)$ term in (21) we have, since $T_A(0)$ is constant,

$$\chi(p)\,T_A(0) = T_A(0)\,\chi(p)\,.\,1$$

$$= T_A(0)\frac{1}{2\pi i}\int_L \frac{e^{pt}\chi(p)\,dp}{p},$$

for $t > 0$.

This integral is the same as the preceding integral with t replacing $t-\lambda$, and hence, since $t > 0$, we shall get

$$\chi(p)\,T_A(0) = T_A(0)\left\{1 - \sum_{n=1}^{\infty}\frac{2\eta J_0\left(\frac{\phi_n r}{a}\right)e^{-\frac{h^2\phi_n^2}{a^2}t}}{(\eta^2+\phi_n^2)\,J_0(\phi_n)}\right\}. \tag{23}$$

We have therefore finally from (21), (22), and (23)

$$\mathrm{T} = \mathrm{T_M}(t) - \int_0^t \frac{d\{\mathrm{T_M}(\lambda) - \mathrm{T_A}(\lambda)\}}{d\lambda}\left\{1 - \Sigma u_n \mathrm{J_0}\left(\frac{\phi_n r}{a}\right)e^{-\beta_n(t-\lambda)}\right\}d\lambda + \mathrm{T_A}(0)\left\{1 - \Sigma u_n \mathrm{J_0}\left(\frac{\phi_n r}{a}\right)e^{-\beta_n t}\right\}$$

$$= \mathrm{T_A}(t) + \sum_{n=1}^{\infty} u_n \mathrm{J_0}\left(\frac{\phi_n r}{a}\right)\mathrm{Q}_n(t), \tag{24}$$

where

$$\left.\begin{aligned} a &= \text{radius of cylinder} \\ u_n &= \frac{2\eta}{(\eta^2+\phi_n^2)\,\mathrm{J_0}(\phi_n)} \\ \beta_n &= \frac{h^2\phi_n^2}{a^2} \\ \eta &= \frac{\mathrm{E}a}{\kappa} \\ \mathrm{Q}_n(t) &= \int_0^t e^{-\beta_n(t-\lambda)}\frac{d\{\mathrm{T_M}(\lambda)-\mathrm{T_A}(\lambda)\}}{d\lambda}\,d\lambda - \mathrm{T_A}(0)\,e^{-\beta_n t} \end{aligned}\right\}, \tag{25}$$

and the ϕ_n's are the positive roots of

$$\eta \mathrm{J_0}(\phi) = \phi \mathrm{J_1}(\phi). \tag{26}$$

The interchange of the order of summation and integration is legitimate since the series is uniformly convergent for all values of λ in the range of integration.

The expression for T given by (24) may be seen to satisfy the equations (15), (16), and (17) by direct substitution, equation (17) being satisfied by virtue of the relation

$$1 = \Sigma u_n \mathrm{J_0}\left(\frac{\phi_n r}{a}\right) \qquad (0 \leqslant r \leqslant a).$$

The average temperature $\overline{\mathrm{T}}$ of the mass at any time is given by

$$\overline{\mathrm{T}} = \frac{1}{\pi a^2}\int_0^a 2\pi r \mathrm{T}\,dr,$$

whence, since term by term integration is legitimate, we have from (24)

$$\overline{\mathrm{T}} = \mathrm{T_A}(t) + \sum_{n=1}^{\infty}\frac{2u_n \mathrm{J_1}(\phi_n)}{\phi_n}\mathrm{Q}_n(t). \tag{27}$$

While equation (24) is usually the most convenient form of solution if the temperature is required at several points in the mass, it is not the most suitable form if only the maximum temperature in the mass, which occurs at the centre, is required. Let $\mathrm{T_0}$ be the temperature at the centre, then from (19) we have

$$\left\{\mathrm{I_0}(qa) + \frac{1}{\eta}qa\mathrm{I_1}(qa)\right\}(\mathrm{T_M} - \mathrm{T_0}) = \mathrm{T_M} - \mathrm{T_A}. \tag{28}$$

Now we know that $J_0(\theta)$ can be expressed as an infinite product in the form (WATSON, *loc. cit.*, p. 497),

$$J_0(\theta) = \prod_{n=1}^{\infty} \left\{1 - \frac{\theta^2}{\theta_n^2}\right\},$$

where $\theta_1, \theta_2, \ldots, \theta_n, \ldots$, are the roots of

$$J_0(\theta) = 0.$$

In a similar manner we can write

$$J_0(\phi) - \frac{1}{\eta}\phi J_1(\phi) = \prod_{n=1}^{\infty} \left\{1 - \frac{\phi^2}{\phi_n^2}\right\},$$

where $\phi_1, \phi_2, \ldots, \phi_n, \ldots$, as before denote the roots of (26).

Hence, putting $\phi = iqa$, *i.e.*,

$$\phi^2 = -q^2a^2 = -pa^2/h^2,$$

we obtain

$$I_0(qa) + \frac{1}{\eta} qa I_1(qa) = \prod_{n=1}^{\infty} \left\{1 + \frac{a^2 p}{h^2 \phi_n^2}\right\}$$
$$= \prod_{n=1}^{\infty} \left\{1 + \frac{1}{\beta_n}\frac{\partial}{\partial t}\right\}.$$

Since $T_M - T_0$ is a function only of t we can write d/dt for $\partial/\partial t$ and equation (28) becomes

$$\left[\left(1 + \frac{1}{\beta_1}\frac{d}{dt}\right)\left(1 + \frac{1}{\beta_2}\frac{d}{dt}\right)\ldots\left(1 + \frac{1}{\beta_n}\frac{d}{dt}\right)\ldots\right](T_M - T_0) = T_M - T_A. \tag{29}$$

Now we shall not affect the convergence of the series for T in (24) if we expand $\left(J_0 \dfrac{\phi_n r}{a}\right)$ as a power series in r, and then rearrange so that T is expressed as a power series in r. Let the result of such an operation be

$$T = T_0 + T_2 \frac{r^2}{a^2} + \ldots T_{2n}\left(\frac{r}{a}\right)^{2n} + \ldots, \tag{30}$$

where $T_0, T_2, \ldots, T_{2n}, \ldots$, are functions of t only.

Substituting (30) in (15) and equating coefficients of powers of r we get

$$\left.\begin{aligned} \frac{4h^2}{a^2} T_2 &= \frac{d(T_0 - T_M)}{dt} \\ \frac{4n^2h^2}{a^2} T_{2n} &= \frac{dT_{2n-2}}{dt} \qquad (n = 2, 3, \ldots \infty). \end{aligned}\right\}. \tag{31}$$

We have also from equations (17) and (30)

$$T_0 = T_2 = \ldots = T_{2n} = \ldots = 0, \qquad \text{when } t = 0,$$

and thence from (31) we get

$$\frac{d^n(T_M - T_0)}{dt^n} = 0 \quad (t = 0, \quad n = 0, 1, 2, \ldots). \tag{32}$$

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From equation (29) $T_M - T_0$ may be determined by successive applications of the operation $\dfrac{1}{1+\dfrac{1}{\beta_n}\dfrac{d}{dt}}$, where

$$\left\{\frac{1}{1+\dfrac{1}{\beta_n}\dfrac{d}{dt}}\right\} f(t) = \beta_n \int_0^t e^{-\beta_n(t-\lambda)} f(\lambda)\, d\lambda,$$

the lower limit being taken zero in virtue of equation (32).

For the evaluation of this type of integral, when $f(t)$ is an empirical function, see § 10. It will be noted that the solution (29) involves successive integration while the series solution (24) involves the addition of terms each containing one integral of the above type. The integration difficulties are thus the same for the two types of solution, but the alternative form (29) saves the calculation of the coefficients u_n and the subsequent multiplication and addition which is required in the series solution, and thus in practice it is quicker to use the solution (29) when only the temperature at the centre is required.

§ 4. *Temperature Rise in Plane Slab.* NEWTON'S *Boundary Law at both Faces.*

The slab is assumed to be bounded by the two parallel planes $x = 0$, $x = a$, and there is no heat flow parallel to the planes. The air temperatures and emissivities at the two faces are not necessarily the same, and are denoted by the suffixes 1 and 2 at the faces $x = 0$ and $x = a$ respectively. The additional complications as compared with the cylinder problem arise from the occurrence of two boundaries. Our general equations in this case become

$$h^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} - \frac{dT_M}{\partial t}, \tag{33}$$

$$\kappa \frac{\partial T}{\partial x} = E_1 (T - T_{A_1}) \qquad (x = 0), \tag{34}$$

$$-\kappa \frac{\partial T}{\partial x} = E_2 (T - T_{A_2}) \qquad (x = a), \tag{35}$$

$$T = 0 \quad \text{when } t = 0 \quad (0 \leqslant x \leqslant a). \tag{36}$$

Putting $\dfrac{\partial}{\partial t} = p = h^2 q^2$ we obtain from (33)

$$\frac{\partial^2 T}{\partial x^2} = q^2 (T - T_M),$$

which has the solution

$$T = T_M - A \cosh qx - B \sinh qx. \tag{37}$$

Applying (34) and (35) we obtain

$$-\kappa q B = E_1 (T_M - A - T_{A_1})$$
$$\kappa q (A \sinh qa + B \cosh qa) = E_2 (T_M - A \cosh qa - B \sinh qa - T_{A_2}).$$

Hence putting

$$E_1 a/\kappa = \eta_1, \qquad E_2 a/\kappa = \eta_2,$$

we obtain

$$\left.\begin{aligned} A &= \frac{(\eta_2 \sinh qa + qa \cosh qa)\,\eta_1 (T_M - T_{A_1}) + \eta_2 qa\,(T_M - T_{A_2})}{(\eta_1\eta_2 + q^2a^2) \sinh qa + (\eta_1 + \eta_2)\, qa \cosh qa} \\ B &= \frac{-\eta_2 (\cosh qa + qa \sinh qa)\,\eta_1 (T_M - T_{A_1}) + \eta_2\eta_1 (T_M - T_{A_2})}{(\eta_1\eta_2 + q^2a^2) \sinh qa + (\eta_1 + \eta_2)\, qa \cosh qa} \end{aligned}\right\}. \tag{38}$$

(36), (37), and (38) give the operational solution of the problem and we interpret this solution by a contour integral as in § 3. It may be shown (*cf.* § 9) that the denominator in (38) has no roots except when qa is zero or a pure imaginary and the interpretation then follows in a similar manner to that of the preceding section. We shall therefore give here the final result only of such interpretation, viz.,

$$\begin{aligned} T = \frac{\eta_1 (1 + \eta_2) - \eta_1\eta_2 (x/a)}{\eta_1 + \eta_2 + \eta_1\eta_2} T_{A_1}(t) + \frac{\eta_2 + \eta_1\eta_2 (x/a)}{\eta_1 + \eta_2 + \eta_1\eta_2} T_{A_2}(t) \\ + \sum_{n=1}^{\infty} \left\{\cos \frac{\phi_n x}{a} + \frac{\eta_1}{\phi_n} \sin \frac{\phi_n x}{a}\right\} \{u_n Q_{n_1}(t) + v_n Q_{n_2}(t)\}, \end{aligned} \tag{39}$$

where

$$\left.\begin{aligned} &\frac{u_n}{2\eta_1 (\phi_n^2 + \eta_2^2) \sin \phi_n} = \frac{v_n}{2\eta_2 (\eta_1 + \eta_2)\, \phi_n} \\ &\quad = \frac{1}{\sin \phi_n \{(\phi_n^2 - \eta_1\eta_2)^2 + \phi_n^2 (\eta_1 + \eta_2)(\eta_1 + \eta_2 + 1) + (\eta_1 + \eta_2)\,\eta_1\eta_2\}} \\ &Q_{n_1}(t) = \int_0^t e^{-\beta_n (t-\lambda)} \frac{d\{T_M(\lambda) - T_{A_1}(\lambda)\}}{d\lambda}\, d\lambda - T_{A_1}(0)\, e^{-\beta_n t} \\ &Q_{n_2}(t) = \int_0^t e^{-\beta_n (t-\lambda)} \frac{d\{T_M(\lambda) - T_{A_2}(\lambda)\}}{d\lambda}\, d\lambda - T_{A_2}(0)\, e^{-\beta_n t} \\ &\beta_n = h^2\phi_n^2/a^2 \end{aligned}\right\}, \tag{40}$$

and the ϕ 's are the positive roots of the equation,

$$\cot \phi = \frac{\phi^2 - \eta_1\eta_2}{(\eta_1 + \eta_2)\, \phi}. \tag{41}$$

The average temperature of the mass at any time is given by

$$\overline{T} = \frac{1}{a} \int_0^a T\, dx,$$

whence from (39)

$$\begin{aligned} \overline{T} = \frac{\eta_1 + \eta_1\eta_2/2}{\eta_1 + \eta_2 + \eta_1\eta_2} T_{A_1}(t) + \frac{\eta_2 + \eta_1\eta_2/2}{\eta_1 + \eta_2 + \eta_1\eta_2} T_{A_2}(t) \\ + \sum_{n=1}^{\infty} \frac{\phi_n \sin \phi_n + \eta_1 (1 - \cos \phi_n)}{\phi_n^2} [u_n Q_{n_1}(t) + v_n Q_{n_2}(t)]. \end{aligned} \tag{42}$$

Case $\eta_1 = 0$.—In this case we have no heat flow across $x = 0$ and the solution obtained will thus also apply to the symmetrical case of a slab of thickness $2a$ losing heat equally from both faces, since, by symmetry, we then have no heat flow across the central plane which we may take as $x = 0$.

Hence putting $\eta = 0$ and dropping the suffix 2 we obtain

$$T = T_A(t) + \sum_{n=1}^{\infty} v_n \cos \frac{\phi_n x}{a} Q_n(t), \tag{43}$$

where

$$\left.\begin{aligned} v_n &= \frac{2\eta}{\cos \phi_n \{\phi_n^2 + \eta(\eta+1)\}} \\ \eta &= Ea/\kappa \\ Q_n(t) &= \int_0^t e^{-\beta_n(t-\lambda)} \frac{d\{T_M(\lambda) - T_A(\lambda)\}}{d\lambda} d\lambda - T_A(0) e^{-\beta_n t} \\ \beta_n &= h^2\phi_n^2/a^2 \end{aligned}\right\}, \tag{44}$$

and ϕ_n are the positive roots of

$$\cot \phi = \phi/\eta.^* \tag{45}$$

The average temperature is given by

$$\bar{T} = T_A(t) + \sum_{n=1}^{\infty} \frac{v_n \sin \phi_n}{\phi_n} Q_n(t). \tag{46}$$

For this case the maximum temperature in the mass occurs at $x = 0$ and we have as in § 3 the alternative form

$$\left[\left(1 + \frac{1}{\beta_1}\frac{d}{dt}\right)\left(1 + \frac{1}{\beta_2}\frac{d}{dt}\right)\dots\right](T_M - T_0) = T_M - T_A, \tag{47}$$

where β_n is given by (44) and (45).

§ 5. *Temperature Rise in Sphere.* NEWTON'S *Boundary Law.*

The surface of the sphere is taken to be $r = a$ and the heat flow is assumed purely radial.

In this case our general equations become

$$h^2 \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r} = \frac{\partial T}{\partial t} - \frac{dT_M}{dt}, \tag{48}$$

$$-\kappa \frac{\partial T}{\partial r} = E(T - T_A) \qquad (r = a), \tag{49}$$

$$T = 0 \qquad \text{when } t = 0 \qquad (0 \leqslant r \leqslant a). \tag{50}$$

The operational solution of (48) is

$$T = T_M - \frac{Aa \sinh qr}{r},$$

* The solutions of equations (45) and (54) are classical. See FOURIER, "Theorie analytique de la Chaleur." Paris, 1822. KIRCHHOFF, "Vorlesungen über die Theorie der Wärme." Leipzig, 1894.

whence using (49) to determine A, and putting

$$Ea/\kappa = \eta,$$

we obtain

$$T = T_M - \frac{\eta a \sinh qr\,(T_M - T_A)}{qa \cosh qa + (\eta - 1) \sinh qa}. \tag{51}$$

This solution may be interpreted operationally in the same manner as in the previous sections, since it may easily be shown (Carslaw, *loc. cit.*) that the denominator in (51) has roots only when qa is zero or a pure imaginary.

The final solution is

$$T = T_A(t) + \Sigma \frac{a}{r} \sin \frac{\phi_n r}{a} u_n Q_n(t), \tag{52}$$

where

$$\left.\begin{aligned} u_n &= \frac{2\eta}{\sin \phi_n \{\eta(\eta - 1) + \phi_n^2\}} \\ \eta &= Ea/\kappa \\ Q_n(t) &= \int_0^t e^{-\beta_n(t-\lambda)} \frac{d\{T_M(\lambda) - T_A(\lambda)\}}{d\lambda} d\lambda - T_A(0)\, e^{-\beta_n t} \\ \beta_n &= h^2 \phi_n^2/a^2 \end{aligned}\right\}, \tag{53}$$

and ϕ_n are the positive roots, excluding zero, of

$$\tan \phi = \frac{\phi}{1 - \eta}.^* \tag{54}$$

The average temperature is given by

$$\overline{T} = \frac{3}{4\pi a^3} \int_0^a T 4\pi r^2\, dr,$$

whence from (52) and (54)

$$\overline{T} = T_A(t) + \Sigma \frac{3\eta \sin \phi_n}{\phi_n^2} u_n Q_n(t). \tag{55}$$

As in the previous sections we have for the maximum temperature in the mass which occurs at the centre the alternative form

$$\left[\left(1 + \frac{1}{\beta_1}\frac{d}{dt}\right)\left(1 + \frac{1}{\beta_2}\frac{d}{dt}\right) \dots\right](T_M - T_0) = T_M - T_A, \tag{56}$$

where β_n is given by (53) and (54).

§ 6. *Temperature Rise in Circular Cylinder Surrounded by another Medium.*

We take the cylinder to be of radius a and assume the flow of heat to be purely radial and the surrounding medium to be non-heat-evolving.

* See footnote p. 444.

In this case we have as our equations,

In heat-evolving medium, $0 \leqslant r \leqslant a$,

$$\left.\begin{aligned} h^2 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} &= \frac{\partial T}{\partial t} - \frac{dT_M}{dt} \\ T = 0 \qquad & \text{when } t = 0 \end{aligned}\right\}. \tag{57}$$

In surrounding medium, $r > a$,

$$\left.\begin{aligned} h_1{}^2 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T_1}{\partial r} &= \frac{\partial T_1}{\partial t} \\ T_1 = D \qquad & \text{when } t = 0 \\ T_1 \to D \qquad & \text{as} \quad r \to \infty \end{aligned}\right\}. \tag{58}$$

At boundary, $r = a$,

$$\left.\begin{aligned} T &= T_1 \\ -\kappa \frac{\partial T}{\partial r} &= -\kappa_1 \frac{\partial T_1}{\partial r} \end{aligned}\right\}. \tag{59}$$

Putting $\partial/\partial t = p = h^2 q^2$ and $h = \gamma h_1$ we obtain from (57) and (58)

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} = q^2 (T - T_M)$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T_1}{\partial r} = \gamma^2 q^2 (T_1 - D).$$

The solutions of these equations which satisfy the conditions that T is finite at $r = 0$ and that $T_1 \to D$ as $r \to \infty$ are given by

$$T = T_M - A I_0 (qr)$$

$$T_1 = D + B K_0 (\gamma q r),$$

where K_0 is Macdonald's Bessel function (Watson, *loc. cit.*, p. 78), and q is taken to have a positive real part.

Applying (59)

$$\left.\begin{aligned} T_M - A I_0 (qa) &= D + B K_0 (\gamma q a) \\ A \kappa q I_1 (qa) &= B \kappa_1 \gamma q K_1 (\gamma q a) \end{aligned}\right\},$$

whence if $\gamma \kappa_1/\kappa = \eta_1$

$$A = \frac{\eta_1 K_1 (\gamma q a) (T_M - D)}{\eta_1 K_1 (\gamma q a) I_0 (qa) + K_0 (\gamma q a) I_1 (qa)}$$

$$B = \frac{I_1 (qa) (T_M - D)}{\eta_1 K_1 (\gamma q a) I_0 (qa) + K_0 (\gamma q a) I_1 (qa)},$$

and thence

$$\mathrm{T} = \mathrm{T_M} - \frac{\eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q r)\, (\mathrm{T_M} - \mathrm{D})}{\eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q a) + \mathrm{K}_0 (\gamma q a)\, \mathrm{I}_1 (q a)} \tag{60}$$

$$\mathrm{T}_1 = \mathrm{D} + \frac{\mathrm{I}_1 (q a)\, \mathrm{K}_0 (\gamma q r)\, (\mathrm{T_M} - \mathrm{D})}{\eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q a) + \mathrm{K}_0 (\gamma q a)\, \mathrm{I}_1 (q a)}. \tag{61}$$

Consider first of all the interpretation of the second $\mathrm{T_M}$ term in equation (60). We use, as before, the relation

$$\mathrm{T_M}(t) = \int_0^\infty \frac{d\mathrm{T_M}(\lambda)}{d\lambda}\, \mathsf{H}(t-\lambda)\, d\lambda,$$

and we have then to interpret

$$\frac{\eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q r)}{\eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q a) + \mathrm{K}_0 (\gamma q a)\, \mathrm{I}_1 (q a)}\, \mathsf{H}(t-\lambda) = \frac{1}{2\pi i}\int_{\mathrm{L}} \frac{e^{p(t-\lambda)}\, \eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q r)\, dp}{\{\eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q a) + \mathrm{K}_0 (\gamma q a)\, \mathrm{I}_1 (q a)\}\, p}, \tag{62}$$

by Bromwich's rule.

In this the integrand is not an even function of q and has a branch point at the origin, and will therefore in general be double valued. In our case, however, we restrict q to have a positive real part by the condition

$$\pi \geqslant \arg p \geqslant -\pi,$$

and the integrand will then be single valued, but not differentiable at points on negative real axis where $\arg p$ is discontinuous.

Now it may be shown (*cf.* § 9) that $\eta_1 \mathrm{K}_1 (\gamma q a)\, \mathrm{I}_0 (q a) + \mathrm{K}_0 (\gamma q a)\, \mathrm{I}_1 (q a)$ has no zeros when the real part of $qa \geqslant 0$. Hence the only possible pole of integrand in (62) is at the origin.

(i) $t - \lambda < 0$.—In this case consider the integral of the above integrand taken round the contour formed by part of L and a large semicircle C, radius R, drawn to right of L. Then since the integrand is regular within and on the contour we have from Cauchy's theorem that the contour integral $= 0$. If we now let $\mathrm{R} \to \infty$ the integral round C tends to zero by Jordan's lemma and we obtain the result :—

$$\int_{\mathrm{L}} = 0 \qquad (t < \lambda). \tag{63}$$

(ii) $t - \lambda > 0$.—In this case we cannot proceed as in § 3 by using a contour formed by part of L and a large semicircle C′ to left of L, since the integrand is not differentiable at points within the contour which lie also on the negative real axis and hence the theorem of residues cannot be applied. We consider instead the contour ABDEFGHA shown in fig. 2. AB is part of L ; BD, HA, are portions of circle of large radius R and centre at origin ; EFG is portion of circle centre origin and of small radius ε ; DE and HG are lines parallel to, and at a short distance on either side of, the negative real axis.

On and within this contour the integral has no poles and is regular and hence by Cauchy's theorem the integral round contour $= 0$.

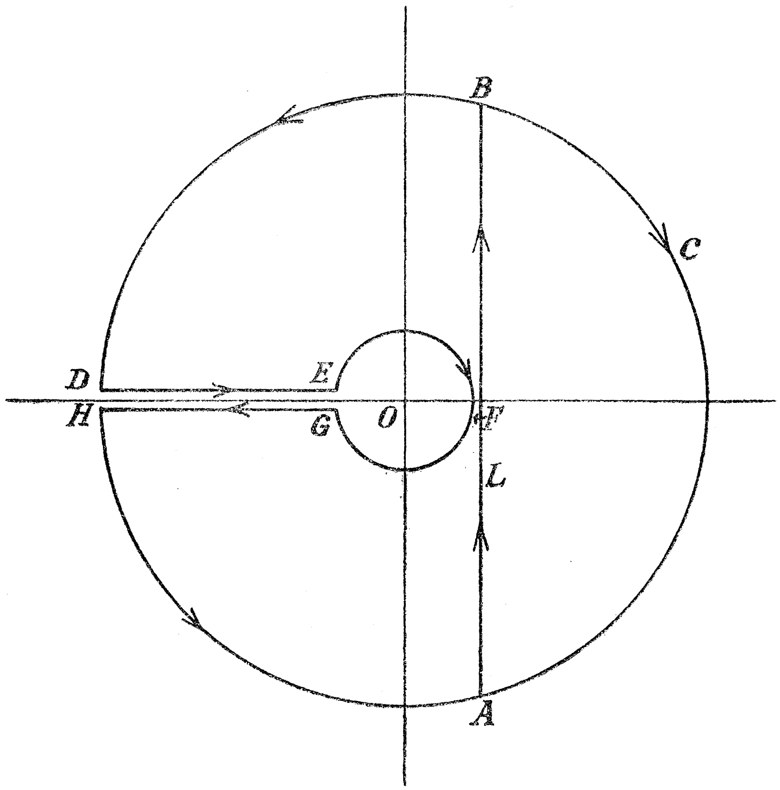

FIG. 2.

We now consider the limit as $R \to \infty$, $\varepsilon \to 0$, and DE, HG become infinitely close to the negative real axis. By JORDAN's lemma the integrals along BD and HA $\to 0$ as $R \to \infty$ and hence we have in limit

$$\int_L = \mathrm{Lt}\left[\int_{ED} - \int_{GH} + \int_{GFE}\right]. \tag{64}$$

Now on GFE

$$\left.\begin{aligned} p &= \varepsilon e^{i\theta} \\ hq &= \varepsilon^{\frac{1}{2}} e^{i\theta/2} \end{aligned}\right\},$$

and in the limit θ goes from $-\pi$ to π.

Now as $\varepsilon \to 0$, $q \to 0$,

$$\left.\begin{aligned} I_0(qr) &\sim I_0(qa) \sim 1 \\ I_1(qa) &\sim \frac{qa}{2} \end{aligned}\right\},$$

$$\left.\begin{aligned} K_0(\gamma qa) &\sim -\log(qa) \\ K_1(\gamma qa) &\sim \frac{1}{\gamma qa} \end{aligned}\right\},$$

therefore

$$\int_{GFE} \sim \int_{-\pi}^{\pi} \frac{i\,d\theta}{\{1 + O(\varepsilon \log \varepsilon)\}}$$

$$\to 2\pi i \text{ as } \varepsilon \to 0.$$

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Now on ED we can write in the limit

$$\left.\begin{aligned} p &= \frac{h^2\xi^2}{a^2} e^{i\pi} \\ qa &= i\xi \end{aligned}\right\} \quad \text{where } \xi \text{ is real and } \geqslant 0.$$

While along GH we can write in the limit

$$\left.\begin{aligned} p &= \frac{h^2\xi^2}{a^2} e^{-i\pi} \\ qa &= -i\xi \end{aligned}\right\} \quad \xi \text{ real} \geqslant 0.$$

Thus the integrals along ED and GH will be conjugate complex functions and thus the difference of the integrals equals twice the imaginary part of the integral along ED.

Now we have (Watson, *loc. cit.*, pp. 73–80)

$$\left.\begin{aligned} \mathrm{K}_0(i\gamma\xi) &= -\tfrac{1}{2}\pi i \{\mathrm{J}_0(\gamma\xi) - i\mathrm{Y}_0(\gamma\xi)\} \\ \mathrm{K}_1(i\gamma\xi) &= -\tfrac{1}{2}\pi \{\mathrm{J}_1(\gamma\xi) - i\mathrm{Y}_1(\gamma\xi)\} \end{aligned}\right\}$$

$$\left.\begin{aligned} \mathrm{I}_0(i\xi) &= \mathrm{J}_0(\xi) \\ \mathrm{I}_1(i\xi) &= i\mathrm{J}_1(\xi) \end{aligned}\right\}.$$

Hence on ED the integrand of (62) becomes

$$\frac{e^{-\frac{h^2\xi^2}{a^2}} \eta_1 \{\mathrm{J}_1(\gamma\xi) - i\mathrm{Y}_1(\gamma\xi)\} \mathrm{J}_0(\xi r/a)}{p\,[\{\eta_1\mathrm{J}_0(\xi)\mathrm{J}_1(\gamma\xi) - \mathrm{J}_1(\xi)\mathrm{J}_0(\gamma\xi)\} + i\{\mathrm{J}_1(\xi)\mathrm{Y}_0(\gamma\xi) - \eta_1\mathrm{J}_0(\xi)\mathrm{Y}_1(\gamma\xi)\}]},$$

and

$$\mathrm{Lt}\left\{\int_{\mathrm{ED}} - \int_{\mathrm{GH}}\right\} = 2\,\mathrm{Lt}\ \mathrm{Imag}\int_{\mathrm{ED}}$$

$$= 2i\int_0^\infty \frac{e^{-\frac{h^2\xi^2}{a^2}(t-\lambda)} \eta_1 \mathrm{J}_1(\xi)\mathrm{J}_0\left(\frac{\xi r}{a}\right)[\mathrm{J}_0(\gamma\xi)\mathrm{Y}_1(\gamma\xi) - \mathrm{J}_1(\gamma\xi)\mathrm{Y}_0(\gamma\xi)]\,2\,d\xi}{\xi \mathrm{F}(\xi)},$$

where

$$\mathrm{F}(\xi) = \{\eta_1\mathrm{J}_0(\xi)\mathrm{J}_1(\gamma\xi) - \mathrm{J}_1(\xi)\mathrm{J}_0(\gamma\xi)\}^2 + \{\eta_1\mathrm{J}_0(\xi)\mathrm{Y}_1(\gamma\xi) - \mathrm{J}_1(\xi)\mathrm{Y}_0(\gamma\xi)\}^2.$$

Now we have (Watson, p. 77)

$$\mathrm{J}_0(\gamma\xi)\mathrm{Y}_1(\gamma\xi) - \mathrm{J}_1(\gamma\xi)\mathrm{Y}_0(\gamma\xi) = -\frac{2}{\pi\gamma\xi}.$$

We therefore have finally from (63) and (64)

$$\frac{1}{2\pi i}\int_{\mathrm{L}} = 0 \qquad (t < \lambda)$$

$$= 1 - \frac{4\eta_1}{\pi^2\gamma}\int_0^\infty \frac{e^{-\frac{h^2\xi^2}{a^2}(t-\lambda)} \mathrm{J}_1(\xi)\mathrm{J}_0\left(\frac{\xi r}{a}\right) d\xi}{\xi^2 \mathrm{F}(\xi)} \qquad (t > \lambda).$$

We have still to interpret the D term in (60). Since D is a constant we can interpret directly as a contour integral and this contour integral is the same as (62) except that t replaces $t - \lambda$.

The interpretation of (60) thus becomes

$$T = T_M(t) - \int_0^t \frac{dT_M(\lambda)}{d\lambda}\left\{1 - \frac{4\eta_1}{\pi^2\gamma}\int_0^\infty \frac{e^{-\frac{h^2\xi^2}{a^2}(t-\lambda)} J_1(\xi) J_0\left(\frac{\xi r}{a}\right) d\xi}{\xi^2 F(\xi)}\right\} d\lambda$$

$$+ D\left\{1 - \frac{4\eta_1}{\pi^2\gamma}\int_0^\infty \frac{e^{-\frac{h^2\xi^2}{a^2}t} J_1(\xi) J_0\left(\frac{\xi r}{a}\right) d\xi}{\xi^2 F(\xi)}\right\}.$$

Since the ξ integral is uniformly convergent for all values of $t - \lambda$ we can change the order of the λ and ξ integrations and obtain after rearrangement

$$T = D + \int_0^\infty u(\xi) J_0\left(\frac{\xi r}{a}\right) Q_1(\xi, t)\, d\xi, \tag{65}$$

where

$$\left.\begin{aligned} a &= \text{radius of cylinder} \\ u(\xi) &= \frac{4\eta_1 J_1(\xi)}{\pi^2\gamma\xi^2 F(\xi)} \\ F(\xi) &= \{\eta_1 J_0(\xi) J_1(\gamma\xi) - J_1(\xi) J_0(\gamma\xi)\}^2 \\ &\qquad + \{\eta_1 J_0(\xi) Y_1(\gamma\xi) - J_1(\xi) Y_0(\gamma\xi)\}^2 \\ \eta_1 &= h\kappa_1/h_1\kappa \\ \gamma &= h/h_1 \\ Q_1(\xi, t) &= \int_0^t e^{-\frac{h^2\xi^2}{a^2}(t-\lambda)} \frac{dT_M(\lambda)}{d\lambda}\, d\lambda - De^{-\frac{h^2\xi^2}{a^2}t} \end{aligned}\right\}. \tag{66}$$

In a similar way we interpret (61) for T_1 and obtain

$$T_1 = D + \int_0^\infty \left\{v_1(\xi) J_0\left(\frac{\gamma\xi r}{a}\right) - v_2(\xi) Y_0\left(\frac{\gamma\xi r}{a}\right)\right\} Q_1(\xi, t)\, d\xi, \tag{67}$$

where

$$\left.\begin{aligned} v_1(\xi) &= \frac{2J_1(\xi)\{J_1(\xi) Y_0(\gamma\xi) - \eta_1 J_0(\xi) Y_1(\gamma\xi)\}}{\pi\xi F(\xi)} \\ v_2(\xi) &= \frac{2J_1(\xi)\{J_1(\xi) J_0(\gamma\xi) - \eta_1 J_0(\xi) J_1(\gamma\xi)\}}{\pi\xi F(\xi)} \end{aligned}\right\}. \tag{68}$$

The average temperature of the mass is given by

$$\bar{T} = D + \int_0^\infty \frac{2u(\xi) J_1(\xi)}{\xi} Q_1(\xi, t)\, d\xi. \tag{69}$$

The preceding solution (65) for T corresponds to the infinite series solutions of the previous sections, but in the present case we cannot present the solution in the alternative form as in §§ 3, 4 and 5.

As stated in § 2 the above solution can be easily extended by means of the transformation (12) to cover the more general case where both media are heat-evolving.

§ 7. *Temperature Rise in Plane Slab, one face bounded by another Medium, other Face obeying* NEWTON'S *Law.*

We take the slab to be bounded by the planes $x = 0$, $x = a$, and assume no heat flow parallel to the plane boundaries. At the face $x = 0$ NEWTON'S law holds while at $x = a$ the slab is bounded by a non-heat-evolving medium.

In this case our general equations become :—

In heat-evolving medium, $0 \leqslant x \leqslant a$,

$$\left.\begin{aligned} h^2 \frac{\partial^2 T}{\partial x^2} &= \frac{\partial T}{\partial t} - \frac{dT_M}{dt} \\ T = 0 \quad &\text{when } t = 0 \end{aligned}\right\} . \tag{70}$$

In bounding medium, $x > a$,

$$\left.\begin{aligned} h_1^2 \frac{\partial^2 T_1}{\partial x^2} &= \frac{\partial T_1}{\partial t}, \\ T_1 = D \quad &\text{when } t = 0 \\ T_1 \to D \quad &\text{as } x \to \infty \end{aligned}\right\} . \tag{71}$$

At boundary $x = a$,

$$\left.\begin{aligned} T &= T_1 \\ -\kappa \frac{\partial T}{\partial x} &= -\kappa_1 \frac{\partial T_1}{\partial x} \end{aligned}\right\} . \tag{72}$$

At boundary $x = 0$,

$$\kappa \frac{\partial T}{\partial x} = E(T - T_A). \tag{73}$$

Putting $\partial/\partial t = p = h^2 q^2 = \gamma^2 h_1^2 q^2$ we obtain from (70) and (71)

$$\frac{\partial^2 T}{\partial x^2} = q^2 (T - T_M),$$

$$\frac{\partial^2 T_1}{\partial x^2} = \gamma^2 q^2 (T_1 - D),$$

whence we have the solutions

$$\left.\begin{aligned} T &= T_M - A \cosh qx - B \sinh qx \\ T_1 &= D + Ce^{-\gamma q x} \end{aligned}\right\}, \tag{74}$$

where the real part of q is positive, since $T_1 \to D$ as $x \to \infty$. From (72) we now have

$$\left.\begin{aligned} T_M - A \cosh qa - B \sinh qa &= D + Ce^{-\gamma q a} \\ \kappa q (A \sinh qa + B \cosh qa) &= \kappa_1 C \gamma q e^{-\gamma q a} \end{aligned}\right\}$$

while from (73) we have

$$-\kappa B q = E(T_M - A - T_A).$$

Putting

$$h\kappa_1/h_1\kappa = \eta_1, \qquad \mathrm{E}a/\kappa = \eta,$$

we obtain

$$\left.\begin{aligned} \mathrm{A} &= \frac{\eta_1 qa\,(\mathrm{T_M}-\mathrm{D}) + \eta\,(\cosh qa + \eta_1 \sinh qa)\,(\mathrm{T_M}-\mathrm{T_A})}{(\eta\eta_1 + qa)\sinh qa + (\eta + \eta_1 qa)\cosh qa} \\ \mathrm{B} &= \frac{\eta\eta_1\,(\mathrm{T_M}-\mathrm{D}) - \eta\,(\sinh qa + \eta_1 \cosh qa)\,(\mathrm{T_M}-\mathrm{T_A})}{(\eta\eta_1 + qa)\sinh qa + (\eta + \eta_1 qa)\cosh qa} \\ \mathrm{C} &= \frac{(qa\sinh qa + \eta\cosh qa)\,e^{\gamma qa}\,(\mathrm{T_M}-\mathrm{D}) - \eta e^{\gamma qa}\,(\mathrm{T_M}-\mathrm{T_A})}{(\eta\eta_1 + qa)\sinh qa + (\eta + \eta_1 qa)\cosh qa} \end{aligned}\right\}. \tag{75}$$

Equations (74) and (75) give the operational solution of the problem and after showing that the denominator in (75) has no roots with the real part of $qa \geqslant 0$ (*cf.* § 9) we can interpret this solution by the same method employed in the preceding section. The final result for T is

$$\mathrm{T} = \mathrm{T_A}(t) + \int_0^\infty \left(\xi\cos\frac{\xi x}{a} + \eta\sin\frac{\xi x}{a}\right)\{u(\xi)\,\mathrm{Q}(\xi, t) + v(\xi)\,\mathrm{Q_1}(\xi, t)\}\,d\xi, \tag{76}$$

where

$$\left.\begin{aligned} a &= \text{thickness of slab} \\ \gamma &= h/h_1 \\ \eta &= \mathrm{E}a/\kappa \\ \eta_1 &= h\kappa_1/h_1\kappa \\ \frac{u(\xi)}{\eta} &= \frac{v(\xi)}{\xi\sin\xi - \eta\cos\xi} \\ &= \frac{2\eta_1}{\pi\xi\,[(\eta\cos\xi - \xi\sin\xi)^2 + \eta_1^2\,(\eta\sin\xi + \xi\cos\xi)^2]} \\ \mathrm{Q}(\xi, t) &= \int_0^t e^{-\frac{h^2\xi^2}{a^2}(t-\lambda)}\,\frac{d\,\{\mathrm{T_M}(\lambda) - \mathrm{T_A}(\lambda)\}}{d\lambda}\,d\lambda - \mathrm{T_A}(0)\,e^{-\frac{h^2\xi^2}{a^2}t} \\ \mathrm{Q_1}(\xi, t) &= \int_0^t e^{-\frac{h^2\xi^2}{a^2}(t-\lambda)}\,\frac{d\mathrm{T_M}(\lambda)}{d\lambda}\,d\lambda - \mathrm{D}\,e^{-\frac{h^2\xi^2}{a^2}t} \end{aligned}\right\}. \tag{77}$$

We obtain also for $\mathrm{T_1}$

$$\mathrm{T_1} = \mathrm{T_A}(t) + \frac{1}{\eta_1}\int_0^\infty \left[\begin{aligned} &(\eta\cos\xi - \xi\sin\xi)\sin\gamma\xi\left(\frac{x}{a} - 1\right) \\ &\qquad + \eta_1\,(\eta\sin\xi + \xi\cos\xi)\cos\gamma\xi\left(\frac{x}{a} - 1\right) \end{aligned}\right] \times \{u(\xi)\,\mathrm{Q}(\xi, t) + v(\xi)\,\mathrm{Q_1}(\xi, t)\}\,d\xi. \tag{78}$$

The average temperature throughout the mass is given by

$$\overline{\mathrm{T}} = \mathrm{T_A}(t) + \int_0^\infty \left\{\sin\xi + \frac{\eta}{\xi}(1 - \cos\xi)\right\}\{u(\xi)\,\mathrm{Q}(\xi, t) + v(\xi)\,\mathrm{Q_1}(\xi, t)\}\,d\xi. \tag{79}$$

As in the preceding section we cannot present the solution in the alternative form as in §§ 3, 4, and 5, but by means of equation (12) we can easily extend the solution to cover the more general case where both media are heat-evolving.

Special Case $\eta = 0$.—In this case we have no heat flow across $x = 0$ and the solution obtained will thus apply to the symmetrical case of a slab of thickness $2a$ bounded at both faces by the same medium.

Letting $\eta \to 0$ we obtain

$$\mathrm{T} = \mathrm{D} + \frac{2\eta_1}{\pi}\int_0^\infty \frac{\cos\frac{\xi x}{a}\sin\xi\,\mathrm{Q}_1(\xi, t)\,d\xi}{\xi\,[\sin^2\xi + \eta_1{}^2\cos^2\xi]}, \tag{80}$$

$$\mathrm{T}_1 = \mathrm{D} + \frac{2}{\pi}\int_0^\infty \frac{\left\{\eta_1\cos\xi\cos\gamma\xi\left(\frac{x}{a} - 1\right) - \sin\xi\sin\gamma\xi\left(\frac{x}{a} - 1\right)\right\}\sin\xi\,\mathrm{Q}_1(\xi, t)\,d\xi}{\xi\,[\sin^2\xi + \eta_1{}^2\cos^2\xi]}, \tag{81}$$

$$\overline{\mathrm{T}} = \mathrm{D} + \frac{2\eta_1}{\pi}\int_0^\infty \frac{\mathrm{Q}_1(\xi, t)\,d\xi}{\xi^2(1 + \eta_1{}^2\cot^2\xi)}. \tag{82}$$

§ 8. *Temperature Rise in Sphere surrounded by another Medium.*

We consider the radial flow of heat in a sphere of radius a surrounded by another homogeneous medium.

The equations in this case are :—

In heat-evolving medium, $0 \leqslant r \leqslant a$,

$$\left.\begin{aligned} h^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial \mathrm{T}}{\partial r} &= \frac{\partial \mathrm{T}}{\partial t} - \frac{d\mathrm{T}_\mathrm{M}}{dt} \\ \mathrm{T} = 0 \quad &\text{when} \quad t = 0 \end{aligned}\right\}. \tag{83}$$

In surrounding medium, $r > a$,

$$h_1{}^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial \mathrm{T}_1}{\partial r} = \frac{\partial \mathrm{T}_1}{\partial t}, \tag{84}$$

$$\mathrm{T}_1 = \mathrm{D} \quad \text{when} \quad t = 0,$$

$$\mathrm{T}_1 \to \mathrm{D} \text{ as } r \to \infty.$$

At boundary $r = a$,

$$\left.\begin{aligned} \mathrm{T} &= \mathrm{T}_1 \\ -\kappa\frac{\partial \mathrm{T}}{\partial r} &= -\kappa_1\frac{\partial \mathrm{T}_1}{\partial r} \end{aligned}\right\}. \tag{85}$$

Putting

$$\frac{\partial}{\partial t} = p = h^2q^2 = \gamma^2h_1{}^2q^2.$$

the solutions of (83) and (84) are

$$\left.\begin{aligned} T &= T_M - \frac{Aa \sinh qr}{r} \\ T_1 &= D + \frac{Ba\, e^{-\gamma qr}}{r} \end{aligned}\right\},$$

where real part of $q > 0$ since $T_1 \to D$ as $r \to \infty$.

Applying (85) and putting $\eta_1 = h\kappa_1/h_1\kappa$ we obtain finally

$$\left.\begin{aligned} T &= T_M - \frac{a}{r}\,\frac{\eta_1(\gamma qa + 1)\sinh qr\,(T_M - D)}{\gamma qa \cosh qa + \sinh qa\,(\eta_1\gamma qa + \eta_1 - \gamma)} \\ T_1 &= D + \frac{a}{r}\,\frac{\gamma\,(qa\cosh qa - \sinh qa)\,e^{-\gamma q(r-a)}\,(T_M - D)}{\gamma qa\cosh qa + \sinh qa\,(\eta_1\gamma qa + \eta_1 - \gamma)} \end{aligned}\right\}.$$

Since the denominator of these expressions has no roots with real part $qa \geqslant 0$ (*cf.* § 9) we interpret these operational solutions by the method used in § 6, and we obtain finally

$$T = D + \frac{a}{r}\int_0^\infty u(\xi)\sin\frac{\xi r}{a}\,Q_1(\xi, t)\,d\xi, \tag{86}$$

where

$$\left.\begin{aligned} a &= \text{radius of sphere} \\ u(\xi) &= \frac{2\eta_1\gamma^2(\sin\xi - \xi\cos\xi)}{\pi\,[\eta_1{}^2\gamma^2\xi^2\sin^2\xi + \{(\eta_1 - \gamma)\sin\xi + \gamma\xi\cos\xi\}^2]} \\ \eta_1 &= h\kappa_1/h_1\kappa \\ \gamma &= h/h_1 \\ Q_1(\xi, t) &= \int_0^t e^{-\frac{h^2\xi^2}{a^2}(t-\lambda)}\,\frac{dT_M(\lambda)}{d\lambda} - D\,e^{-\frac{h^2\xi^2}{a^2}t} \end{aligned}\right\}. \tag{87}$$

We obtain also for T_1

$$T_1 = D + \frac{a}{r}\int_0^\infty u(\xi)\left[\sin\xi\cos\gamma\xi\,\frac{r-a}{a} + \left\{\left(\frac{1}{\gamma} - \frac{1}{\eta_1}\right)\frac{\sin\xi}{\xi} + \frac{1}{\eta_1}\cos\xi\right\}\sin\gamma\xi\,\frac{r-a}{a}\right]Q_1(\xi, t)\,d\xi. \tag{88}$$

For the average temperature of the sphere we find

$$\overline{T} = D + \int_0^\infty \frac{u(\xi)\,3\,(\sin\xi - \xi\cos\xi)\,Q_1(\xi, t)\,d\xi}{\xi^2}. \tag{89}$$

As in the preceding two sections the above solutions may, by means of the transformation (12), be easily extended to the case where both media are heat-evolving.

§ 9. *Mathematical Lemmas.*

In this section we shall now prove some theorems which were deferred in the previous sections.

(*a*) To show that the equation

$$(\theta^2 + \eta_1\eta_2)\sinh\theta + (\eta_1 + \eta_2)\,\theta\cosh\theta = 0, \tag{90}$$

where $\eta_1\eta_2$ are real and $\geqslant 0$ has no roots except when θ is zero or a pure imaginary.

When θ is real all terms in the equation have the same sign and there is no root other than $\theta = 0$.

Let us assume there is a complex root $\theta = \xi + i\phi$ where neither ξ nor ϕ is zero; then there is also another root $\xi - i\phi$ since the left-hand side of (90) is a real function of θ.

Now we have

$$\int_0^1 \sinh kz \sinh lz \, dz = \frac{1}{k^2 - l^2} [k \cosh k \sinh l - l \cosh l \sinh k], \tag{91}$$

where the path of z is along the real axis.

Hence

$$\frac{\eta_1 + \eta_2}{\sinh k \sinh l} \int_0^1 \sinh kz \sinh lz \, dz + 1 = \frac{1}{k^2 - l^2}\left[(\eta_1 + \eta_2)\frac{k \cosh k}{\sinh k} + k^2 - (\eta_1 + \eta_2)\frac{l \cosh l}{\sinh l} - l^2\right]. \tag{92}$$

Now put

$$\left.\begin{aligned} k &= \xi + i\phi \\ l &= \xi - i\phi \end{aligned}\right\}.$$

Then since k and l are roots of (90) the right-hand side of (92) vanishes, while since k and l are conjugate complex numbers the left-hand side of (92) is essentially positive. We have thus arrived at a contradiction and hence the assumption that there are complex roots of (90) is untrue which proves the required theorem.

(*b*) To show that the equation

$$(\eta + \eta_1\theta) \cosh \theta + (\theta + \eta\eta_1) \sinh \theta = 0, \tag{93}$$

where η, η_1 are real and > 0 has no roots with a real part $\geqslant 0$.

Now we have

$$\frac{k(k + \eta\eta_1)}{\eta + \eta_1 k} - \frac{l(l + \eta\eta_1)}{\eta + \eta_1 l} = \frac{(k - l)\{\eta(k + l) + \eta_1 kl + \eta^2\eta_1\}}{(\eta + \eta_1 k)(\eta + \eta_1 l)}.$$

Hence from (91) we have

$$\frac{1}{\sinh k \sinh l} \int_0^1 \sinh kz \sinh lz \, dz + \frac{\eta(k + l) + \eta_1 kl + \eta^2\eta_1}{(k + l)(\eta + \eta_1 k)(\eta + \eta_1 l)} = \frac{1}{k^2 - l^2}\left[\left\{\frac{k \cosh k}{\sinh k} + \frac{k(k + \eta\eta_1)}{\eta + \eta_1 k}\right\} - \left\{\frac{l \cosh l}{\sinh l} + \frac{l(l + \eta\eta_1)}{\eta + \eta_1 l}\right\}\right]. \tag{94}$$

Now let $\theta = \xi + i\phi$ where ξ, ϕ, are real and $\xi > 0$ be a root of (93). Then $0 = \xi - i\phi$ is also a root. Hence putting in (94)

$$\left.\begin{aligned} k &= \xi + i\phi \\ l &= \xi - i\phi \end{aligned}\right\},$$

the right-hand side vanishes by virtue of (93) while the left-hand side is > 0, *i.e.*, we have a contradiction and hence there can be no roots with $\xi > 0$.

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Also if $\xi = 0$ then from (93) by equating real and imaginary parts to zero we obtain

$$\left.\begin{array}{r}\eta \cos \phi - \phi \sin \phi = 0 \\ \eta_1 \phi \cos \phi + \eta_1 \eta \sin \phi = 0\end{array}\right\} \text{ therefore } \eta_1 (\phi^2 + \eta^2) = 0,$$

which is impossible since ϕ is real. Hence we can have no roots with $\xi \geqslant 0$.

(*c*) To show that the equation

$$\gamma\theta \cosh \theta + (\eta_1 \gamma\theta + \eta_1 - \gamma) \sinh \theta = 0, \tag{95}$$

where η_1, γ, are real and > 0 has no roots (except zero) with a real part $\geqslant 0$.

We have from (91)

$$\frac{\gamma}{\sinh k \sinh l} \int_0^1 \sinh kz \sinh lz \, dz + \frac{\eta_1 \gamma}{k + l}$$
$$= \frac{1}{k^2 - l^2} \left[\left\{ \frac{\gamma k \cosh k}{\sinh k} + \eta_1 \gamma k \right\} - \left\{ \frac{\gamma l \cosh l}{\sinh l} + \eta_1 \gamma l \right\} \right]. \tag{96}$$

Hence if $\theta = \xi + i\phi$, $\xi - i\phi$, are roots of (95) with $\xi > 0$ then putting these roots for k and l in (96) we obtain a contradiction since the right-hand side vanishes while the left-hand side is > 0. Hence there are no roots with $\xi > 0$. If $\xi = 0$ then putting $\theta = i\phi$ in (95) we obtain

$$\left.\begin{array}{r}\phi \sin \phi = 0 \\ \gamma\phi \cos \phi + (\eta_1 - \gamma) \sin \phi = 0\end{array}\right\},$$

which are incompatible unless $\phi = 0$. Hence we can have no roots other than zero with a real part $\geqslant 0$.

(*d*) To show that the equation

$$\eta_1 I_0 (\theta) K_1 (\gamma\theta) + I_1 (\theta) K_0 (\gamma\theta) = 0, \tag{97}$$

where η_1, γ are real and > 0 has no roots when the real part of $\theta \geqslant 0$. We have (WATSON, p. 134)

$$\int^z z C_\mu (kz) \bar{C}_\mu (lz) \, dz = \frac{z}{k^2 - l^2} \{k C_{\mu+1} (kz) \bar{C}_\mu (lz) - l C_\mu (kz) \bar{C}_{\mu+1} (lz)\}, \tag{98}$$

where C_μ, $\bar{C}_\mu$ are any cylinder functions.

Putting

$$\left.\begin{array}{r} k = i\alpha \\ l = i\beta \end{array}\right\} C_\mu = \bar{C}_\mu = J_0,$$

we obtain after simplification

$$\int^z z I_0 (\alpha z) I_0 (\beta z) \, dz = \frac{z}{\beta^2 - \alpha^2} \{\beta I_0 (\alpha z) I_1 (\beta z) - \alpha I_0 (\beta z) I_1 (\alpha z)\},$$

whence

$$\int_0^1 z I_0 (\alpha z) I_0 (\beta z) \, dz = \frac{1}{\beta^2 - \alpha^2} \{ \beta I_0 (\alpha) I_1 (\beta) - \alpha I_0 (\beta) I_1 (\alpha) \}, \tag{99}$$

in which z is assumed real.

Similarly by putting

$$\left.\begin{array}{l} k = i\alpha \\ l = i\beta \end{array}\right\} \mathrm{C}_\mu = \overline{\mathrm{C}}_\mu = \mathrm{H}_0^{(1)}$$

we obtain (WATSON, pp. 73, 78)

$$\int^z z\mathrm{K}_0(\alpha z)\,\mathrm{K}_0(\beta z)\,dz = \frac{z}{\beta^2-\alpha^2}[\alpha\mathrm{K}_0(\beta z)\,\mathrm{K}_1(\alpha z) - \beta\mathrm{K}_0(\alpha z)\,\mathrm{K}_1(\beta z)].$$

Now if we take z real and restrict α and β to have real parts > 0 then as $z \to \infty$ the expression above on right $\to 0$ and we therefore have

$$\int_\gamma^\infty z\mathrm{K}_0(\alpha z)\,\mathrm{K}_0(\beta z)\,dz = \frac{\gamma}{\beta^2-\alpha^2}[\beta\mathrm{K}_0(\alpha\gamma)\,\mathrm{K}_1(\beta\gamma) - \alpha\mathrm{K}_0(\beta\gamma)\,\mathrm{K}_1(\alpha\gamma)]. \tag{100}$$

We thus have from (99) and (100)

$$\frac{\eta_1}{\mathrm{I}_1(\alpha)\,\mathrm{I}_1(\beta)}\int_0^1 z\mathrm{I}_0(\alpha z)\,\mathrm{I}_0(\beta z)\,dz + \frac{1}{\gamma\mathrm{K}_1(\alpha\gamma)\,\mathrm{K}_1(\beta\gamma)}\int_\gamma^\infty z\mathrm{K}_0(\alpha z)\,\mathrm{K}_0(\beta z)\,dz$$
$$= \frac{\beta}{\beta^2-\alpha^2}\left\{\frac{\eta_1\mathrm{I}_0(\alpha)}{\mathrm{I}_1(\alpha)} + \frac{\mathrm{K}_0(\alpha\gamma)}{\mathrm{K}_1(\alpha\gamma)}\right\} - \frac{\alpha}{\beta^2-\alpha^2}\left\{\frac{\eta_1\mathrm{I}_0(\beta)}{\mathrm{I}_1(\beta)} + \frac{\mathrm{K}_0(\beta\gamma)}{\mathrm{K}_1(\beta\gamma)}\right\}. \tag{101}$$

Let us now take $\theta = \xi + i\phi$, $\xi - i\phi$, to be roots of (97) with a real part $\xi > 0$.

Since both roots have a positive real part they are permissible values for α and β and we can substitute them in (101).

The right-hand side of (101) then vanishes by virtue of (97) while the left-hand side is > 0 since we have only products of conjugate functions occurring. We have thus arrived at a contradiction by assuming there are roots of (97) having no real part > 0 and there cannot therefore be any such roots.

Let us now assume a root with a real part equal to zero and let us in (97) put $\theta = i\phi$, where ϕ is real, and equate real and imaginary parts of the equation to zero. We then obtain

$$\left.\begin{array}{l} \eta_1\,\mathrm{J}_0(\phi)\,\mathrm{J}_1(\gamma\phi) - \mathrm{J}_1(\phi)\,\mathrm{J}_0(\gamma\phi) = 0 \\ \eta_1\,\mathrm{J}_0(\phi)\,\mathrm{Y}_1(\gamma\phi) - \mathrm{J}_1(\phi)\,\mathrm{Y}_0(\gamma\phi) = 0 \end{array}\right\}, \tag{102}$$

whence eliminating $\mathrm{J}_1(\phi)$,

$$\eta_1\,\mathrm{J}_0(\phi)\,\{\mathrm{J}_1(\gamma\phi)\,\mathrm{Y}_0(\gamma\phi) - \mathrm{J}_0(\gamma\phi)\,\mathrm{Y}_1(\gamma\phi)\} = 0. \tag{103}$$

Now we have (WATSON, p. 77)

$$\mathrm{J}_1(\gamma\phi)\,\mathrm{Y}_0(\gamma\phi) - \mathrm{J}_0(\gamma\phi)\,\mathrm{Y}_1(\gamma\phi) = 2/\pi\gamma\phi, \tag{104}$$

and therefore from (103) we must have $\mathrm{J}_0(\phi) = 0$.

Hence, since J_0 and J_1 have no common zeros, we have from (102) that $\mathrm{J}_0(\gamma\phi) = \mathrm{Y}_0(\gamma\phi) = 0$. This is, however, impossible by virtue of (104), and the original assumption that a purely imaginary root of (97) exists is therefore untrue. We have, therefore, proved that any roots of (97) which exist must have a real part < 0,

§ 10. *Evaluation of Solutions.*

When we attempt to calculate T from one of the preceding series or integral solutions the first difficulty which we encounter is the evaluation of the λ integrals involved in the Q terms.

Since in practice $T_M(t)$ and $T_A(t)$ will be empirical functions of t we cannot perform the integration mathematically unless we first represent $T_M(t)$ and $T_A(t)$ as mathematical functions of such a type that the integral can then be expressed in terms of known functions. For concrete, it has been found impossible to represent $T_M(t)$ by any such function, except as a polynomial or a sum of exponentials, and in such cases the number of terms required to give a sufficiently accurate representation is so great that the labour of calculation is prohibitive.

In such a case, we must, therefore, turn instead to a method of numerical, mechanical, or graphical integration. Now all Q terms involving λ integrals belong to the general type

$$\int_0^t e^{-\alpha(t-\lambda)} \frac{df(\lambda)}{d\lambda} d\lambda + f(0)\, e^{-\alpha t} = f(t) - \alpha \int_0^t e^{-\alpha(t-\lambda)} f(\lambda)\, d\lambda,$$

and we have therefore to consider the direct evaluation of the integral

$$g(t) = \alpha \int_0^t e^{-\alpha(t-\lambda)} f(\lambda)\, d\lambda. \qquad (105)$$

1. The first and most obvious method of evaluation is to multiply the functions $\alpha e^{\alpha\lambda}$ and $f(\lambda)$ for selected values of λ then integrate the product by SIMPSON's rule or similar formula, and finally multiply by $e^{-\alpha t}$.

2. A second, and better, method of numerical integration may be derived by considering the differential equation satisfied by the integral, viz.,

$$\frac{dg}{dt} + \alpha g = \alpha f, \qquad (106)$$

whence

$$g = \alpha \int_0^t (f - g)\, d\lambda.$$

Divide the range of integration into equal time intervals τ and let the value of g after time $n\tau$ be g_n. Then

$$g_{n+1} - g_n = \alpha \int_{n\tau}^{(n+1)\tau} (f - g)\, d\lambda.$$

Now in the small interval τ we may approximate to g by a parabola and therefore by equation (106) to $f - g$ by a straight line whence

$$g_{n+1} - g_n = \frac{\alpha\tau}{2}(f_{n+1} - g_{n+1}) + \frac{\alpha\tau}{2}(f_n - g_n),$$

therefore

$$f_{n+1} - g_{n+1} = (f_n - g_n)\frac{2 - \alpha\tau}{2 + \alpha\tau} + \frac{2}{2 + \alpha\tau}(f_{n+1} - f_n).$$

This relation gives us a very simple means by which $f - g$ and therefore g can be calculated for successive intervals and by making these intervals as small as we please we may increase the accuracy to any required extent. This method is quicker than the preceding one, for the same degree of accuracy, since we may first tabulate $f_{n+1} - f_n$, multiply through by the constant factor $2/(2 + \alpha\tau)$, and then any further multiplication is by the other constant factor $\frac{2 - \alpha\tau}{2 + \alpha\tau}$. Using tables or a multiplying machine such repeated multiplication is much more easily done than the corresponding multiplication of pairs of different numbers which is required in method 1.

3. A graphical method of integration will now be developed from the differential equation (106). This equation may be written

$$\frac{dg}{dt} = \alpha (f - g). \tag{107}$$

Now suppose g and f to be plotted as in fig. 3, with the origin of t for the g curve taken distance $1/\alpha$ to the left of the origin of t for the f curve.

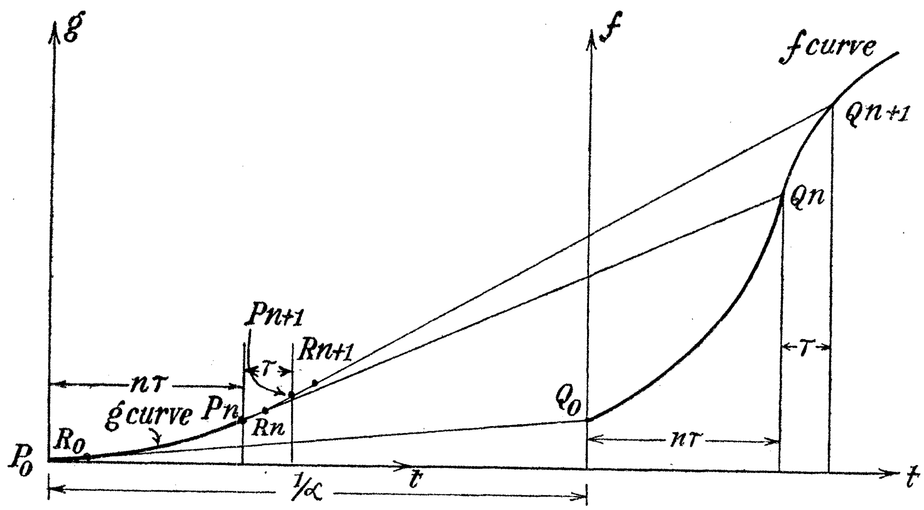

Fig. 3.

Let P_n, Q_n, be points on g and f curves respectively, at the time $t = n\tau$. Then by equation (107) the tangent to g at P_n will pass through Q_n. Similarly the tangent to g at P_{n+1} will pass through Q_{n+1}. Let these two tangents meet at R_n. Now if τ is small we have, correct to second order in τ, that abscissa of R_n is $(n + \frac{1}{2})\,\tau$. Hence, assuming this is true, *i.e.*, neglecting powers of τ of hither order than second, we have the following graphical construction.

Firstly, take origin P_0 of g at a distance $1/\alpha$ to left of origin Q_0 of f. Secondly, draw P_0R_0 where R_0 lies on $t = \frac{1}{2}\tau$ so that P_0R_0 produced passes through Q_0. Thirdly, having obtained R_0 we now proceed by the general construction for obtaining R_{n+1} from R_n, viz., draw $R_n\,R_{n+1}$ so that produced it would pass through Q_{n+1}.

We thus obtain the points R_0, R_1, . . ., and by taking the point P_n where $R_{n-1} R_n$ cuts $t = n\tau$ we obtain the points P_1, P_2, . . ., lying on the required g curve.

The relative accuracy obtained with varying sizes of τ may be judged in practice by the relative smoothness of the envelope formed by the lines P_0R_0, R_0R_1, ... This method has been used for integrals occurring in the specific problem of temperature rise in concrete and it has been found that quite smooth envelope curves can be obtained without having to use an unreasonably small value of τ. The above construction involves only a straight edge and affords a very speedy method of solving the integral (105) or the differential equation (106). While realizing that this graphical method can never be made as accurate as the corresponding numerical method 2 which is based on the same approximation, direct comparison of the values obtained for the same integral by the two methods, has shown that, for engineering applications, the gain in accuracy afforded by the numerical method is usually negligible.

4. Machines have been developed* for solving the integral (105) mechanically, but unless such a machine is already to hand, the expense of purchasing or making will in general more than counterbalance any advantage such a machine might possess over the methods 2 and 3.

Having evaluated the λ integrals, the further computation involved in the series solutions of §§ 3, 4, and 5 presents no difficulty.

Turning now to the alternative form of solution given in §§ 3, 4, and 5, let us consider equation (29) as a typical example.

Now equation (29) may be written in the form of an infinite number of simultaneous equations as follows :—

$$\left(1 + \frac{1}{\beta_1}\frac{d}{dt}\right) g_1 = T_M(t) - T_A(t)$$

$$\left(1 + \frac{1}{\beta_n}\frac{d}{dt}\right) g_n = g_{n-1} \qquad n = 2 \text{ to } \infty$$

$$g_n \to T_M - T_o \qquad \text{as } n \to \infty .$$

We thus see that since $T_M(t) - T_A(t)$ is known g_1 can be obtained directly by the graphical method, and thence by successive application of the method we can derive g_n. Thus by taking n as large as we please we can obtain as close an approximation to $T_M - T_0$ as we desire, in the same way as with the series solution we obtain our approximation to T by stopping after the nth term. In practice we stop when the difference between g_n and g_{n+1} is negligible which in the cases which have been evaluated for concrete occurs for $n = 3$ to 4.

The only remaining point in the evaluation of any of the solutions concerns the ξ integrations occurring in §§ 6, 7, and 8, which must, of course, be performed by means of SIMPSON's rule or a similar quadrature formula. For any integral of § 6 this can be done in the usual way, provided due regard is paid to the oscillatory character of the

* BATEMAN, " Differential Equations," chap. 11 (1926).

integrand, and on this account it is probably best to calculate separately the contributions given by positive and negative values respectively of the integrand. For the ξ integrals of §§ 7 and 8 it is better to avoid this complication by using the special quadrature formula for trigonometric integrals developed by Filon.*

§ 11. *Summary.*

The general equations of heat flow in a medium evolving heat at a rate constant throughout the medium but varying with time are first derived and certain general results thence deduced. Solutions of these equations are then obtained for what are probably the six most important cases of one-dimensional flow. In these solutions the effect of an initial gradient has not been included since it can be simply taken into account by the addition of an auxiliary solution which satisfies the heat conduction equation for an ordinary medium and is therefore of well-known form.†

An additional complexity which has not been considered here occurs when the rate of heat evolution varies with the temperature. In such a case the term ε in equation (1) depends on T as well as t and it has not so far been found possible to obtain a mathematical solution for this case.

The general theory given here has been elsewhere applied with reasonable success to temperature rise in reacting concrete,‡ and since concrete possesses a temperature coefficient of velocity of reaction, the last-mentioned difficulty occurs, but has been satisfactorily surmounted by means of an empirical correction.

* 'Proc. Roy. Soc. Edin.,' vol. 49, p. 38 (1928).

† Carslaw, *loc. cit.*

‡ Davey and Fox, "Temperature Rise in Hydrating Concrete," 'Building Research Technical Paper No. 15'. London, 1933 (H.M. Stationery Office).